Fourier Analysis 03-05.

Review.

§ 3.5 A cts function with diverging Fourier series.

Let us start from a special function:

$$f(x) = \begin{cases} i(\pi - x) & \text{if } 0 \leq x \leq \pi \\ i(-\pi - x) & \text{if } -\pi \leq x < 0 \end{cases}$$

It is an odd function except at $x=0$.

It has the following Fourier series

$$f(x) \sim \sum_{n \neq \infty} \frac{1}{n} e^{inx} \quad \text{on } [-\pi, \pi].$$

Write for $N \in \mathbb{N}$,

$$f_N(x) = \sum_{\substack{-N \leq n \leq N \\ n \neq 0}} \frac{1}{n} e^{inx}$$

$$\tilde{f}_N(x) = \sum_{n=-N}^{-1} \frac{1}{n} e^{inx}$$

Lemma 1. (1) $\exists\ M > 0$ such that

$$|f_N(x)| \leq M \quad \text{for all } N \in \mathbb{N},\ x \in [-\pi, \pi]$$

(2) $|\tilde{f}_N(0)| \geq \log N$.

Pf. We first prove (1). Consider the Abel mean of f,

$$A_r(f)(x) = \sum_{n\neq 0} \frac{r^{|n|}}{n} e^{inx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) f(y)\, dy$$

where $0 \leq r < 1$.

$$|A_r(f)(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) |f(y)|\, dy$$

$$\leq \sup_{z\in[-\pi,\pi]} |f(z)| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y)\, dy$$

$$= \|f\|_\infty < \infty$$

Notice that

$$|f_N(x) - A_r(f)(x)| \leq \Big| f_N(x) - \sum_{0<|n|\leq N} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$+ \Big| \sum_{|n|\geq N+1} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$= \Big| \sum_{0<|n|\leq N} \frac{(1-r^{|n|})}{n} e^{inx} \Big|$$

$$+ \Big| \sum_{|n|\geq N+1} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$\leq \sum_{0<|n|\leq N} \frac{1-r^{|n|}}{|n|} + \sum_{|n|\geq N+1} \frac{r^{|n|}}{|n|}$$

$$= 2\sum_{n=1}^{N} \frac{1-r^n}{n} + 2\cdot \sum_{n\geq N+1} \frac{r^n}{n}$$

(Notice that $\frac{1-r^n}{n} = \frac{(1-r)(1+r+\cdots+r^{n-1})}{n} < 1-r$

$$\sum_{n\geq N+1} \frac{r^n}{n} \leq \sum_{n\geq N+1} \frac{r^n}{N}$$

$$= \frac{r^{N+1}}{N(1-r)} \quad)$$

Hence

$$|f_N(x) - A_r(f)(x)| \leq 2\cdot N(1-r) + 2\cdot \frac{r^{N+1}}{N(1-r)}$$

$$\leq 2\cdot N(1-r) + \frac{2}{N(1-r)}$$

Taking $r = 1-\frac{1}{N}$, then $N(1-r)=1$ so

$$|f_N(x) - A_r(f)(x)| \leq 4$$

Hence $|f_N(x)| \leq |A_r(f)(x)| + 4$

$$\leq \|f\|_\infty + 4 \leq 2\pi + 4.$$

This proves (1).

Now $|\widehat{f_N}(0)| = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$

$$\geq \sum_{k=1}^{N} \int_k^{k+1} \frac{1}{x}\, dx$$

$$\geq \sum_{k=1}^{N} (\log(k+1) - \log k) = \log(N+1) > \log N.$$

$\square$

Now for each $N \in \mathbb{N}$, define

$$P_N(x) = e^{i2Nx} \cdot f_N(x) = \sum_{\substack{n=N \\ n \neq 2N}}^{3N} \frac{1}{n-2N} e^{inx}$$

$$\tilde{P}_N(x) = e^{i2Nx} \tilde{f}_N(x) = \sum_{n=N}^{2N-1} \frac{1}{n-2N} e^{inx}$$

Define a sequence of integers $(N_k)_{k=1}^{\infty}$ and a sequence of positive numbers $(\alpha_k)_{k=1}^{\infty}$ such that

(i) $N_{k+1} > 3N_k$ for all k

(ii) $\sum_{k=1}^{\infty} \alpha_k < \infty$.

(iii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

(e.g. we can take $N_k = 4^{4^k}$, $\alpha_k = 3^{-k}$.)

Define

$$g(x) = \sum_{k=1}^{\infty} \alpha_k \cdot P_{N_k}(x)$$

Since $|P_{N_k}(x)| = |f_{N_k}(x)| \leq M$

So the series converges absolutely and g is cts.

We would like to show that

$$S_N g(0) \not\to g(0).$$

<u>Lemma 2</u>: Let $n \in \mathbb{Z}$. Then

$$\hat{g}(n) = \begin{cases} \dfrac{\alpha_k}{n - 2N_k} := \alpha_k \widehat{P_{N_k}}(n) & \text{if } N_k \leq n \leq 3N_k \text{ but } n \neq 2N_k \\ 0 & \text{otherwise} \end{cases}$$

Pf. Fix $n \in \mathbb{N}$. Let $\varepsilon > 0$. $\exists$ $L \in \mathbb{N}$ such that

$$\left| g(x) - \sum_{j=1}^{\ell} \alpha_j P_{N_j}(x) \right| < \varepsilon \quad \text{if } \ell \geq L$$

for all $x \in [-\pi, \pi]$

Then

$$\left| \hat{g}(n) - \sum_{j=1}^{\ell} \alpha_j \widehat{P_{N_j}}(n) \right| < \varepsilon. \quad (*)$$

Keep in mind that $\widehat{P_{N_j}}(n) = 0$ if $n \notin [N_j, 3N_j]$

Hence if $n \notin \bigcup_{j=1}^{\infty} [N_j, 3N_j]$, then

$$\sum_{j=1}^{\ell} \alpha_j \widehat{P_{N_j}}(n) = 0 \quad \text{for all } \ell.$$

By $(*)$, we see that $|\hat{g}(n)| < \varepsilon$, Since ε is arbitrarily given, we have $\widehat{g}(n) = 0$.

Next assume $n \in [N_k, 3N_k]$ for some k.

Then for $l \geq k$.

$$\sum_{j=1}^{l} \alpha_j \widehat{P_{N_j}}(n) = \alpha_k \widehat{P_{N_k}}(n) = \begin{cases} \dfrac{\alpha_k}{n-2N_k} & \text{if } n \neq 2N_k \\ 0 & \text{if } n = 2N_k \end{cases}$$

Again by (*), we get

$$\left| \widehat{g}(n) - \alpha_k \widehat{P_{N_k}}(n) \right| < \varepsilon$$

$$\Rightarrow \quad \widehat{g}(n) = \alpha_k \widehat{P_{N_k}}(n). \qquad \square$$

Let us consider $S_{2N_m}(g)(0)$.

$$g(x) = \underbrace{\underset{[N_1,\,3N_1]}{\alpha_1 P_{N_1}(x)} + \underset{[N_2,\,3N_2]}{\alpha_2 P_{N_2}(x)} + \cdots + \underset{[N_{m-1},\,3N_{m-1}]}{\alpha_{m-1} P_{N_{m-1}}(x)} + \underset{[N_m,\,3N_m]}{\alpha_m P_{N_m}(x)}}_{S_{2N_m}(g)(x)} + \cdots$$

$n = 2N_m$

That is, $S_{2N_m}(g)(x) = \alpha_1 P_{N_1}(x) + \cdots + \alpha_{m-1} P_{N_{m-1}}(x) + \alpha_m \widetilde{P}_{N_m}(x)$

Hence

$$S_{2N_m}(g)(0) = \underbrace{\alpha_1 P_{N_1}(0) + \cdots + \alpha_{m-1} P_{N_{m-1}}(x)}_{(\mathrm{I})} + \underbrace{\alpha_m \widetilde{P}_{N_m}(0)}_{(\mathrm{II})}$$

$$|(\mathrm{I})| \leq \alpha_1 \cdot M + \alpha_2 M + \cdots + \alpha_{m-1} M \leq \left(\sum_{j=1}^{\infty} \alpha_j \right) M < \text{const.}$$

$$|(\mathrm{II})| \geq \alpha_m \cdot \log N_m \to \infty \quad \text{as} \quad m \to \infty$$

Hence $S_{2N_m}(g)(0) \to \infty$ as $m \to \infty$

i.e. $S_n(g)(0)$ diverges ! $\blacksquare$.

Chap 4. Applications of Fourier series.

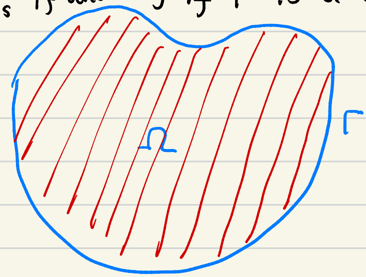
§4.1 Isoperimetric inequality.

Thm 1. Let Γ be a C^1 simple closed curve in $\mathbb{R}^2$.

Then
$$A \leq \frac{\ell^2}{4\pi},$$
where $\ell = \ell(\Gamma)$ is the length of Γ and

A is the area of the region bounded by Γ.

The "$=$" holds if and only if Γ is a circle.



$A = \text{Area}(\Omega)$

$\ell = \text{length}(\Gamma)$

Def: A parametrize curve in $\mathbb{R}^2$ is a mapping
$$\gamma : [a,b] \to \mathbb{R}^2.$$
The image of γ, $\{\gamma(t) : t \in [a,b]\}$

is called a curve, and is denoted by Γ.

We say γ is C^1 if $t \mapsto \gamma(t)$ is C^1 and

$\gamma'(t) \neq 0$ for any $t \in [a,b]$.

Def. We say that $\gamma : [0, \ell] \to \mathbb{R}^2$ is a curve parametrized by arc-length if

$$|\dot{\gamma}(t)| = 1 \quad \forall\ t \in [0, \ell]. \quad (**)$$

$$\left(\Leftrightarrow \quad x'(t)^2 + y'(t)^2 = 1 \quad \text{for } \gamma(t) = (x(t), y(t)) \right)$$

Basic fact: For any C^1 parametrized curve $\gamma = \gamma(t)$, $t \in [a,b]$

$$\text{length of } \Gamma = \int_a^b |\dot{\gamma}(t)|\, dt.$$

If γ is parametrized by arc length,

then $$\int_0^s |\dot{\gamma}(t)|\, dt = \int_0^s 1\, dt = s, \quad \forall\ s \in [0, \ell]$$

Basic fact: Any C^1 curve in the plane allows a parametrization by arc-length.

Pf. Taking a suitable transformation $(x, y) \mapsto (\delta x, \delta y)$ we may assume $\ell(\Gamma) = 2\pi$.

Then we need to show that $A \leq \pi$.

Parametriz Γ by its arc-length, say

$$\gamma = \gamma(t) = (x(t), y(t)), \quad t \in [0, 2\pi]$$

Then $\quad x'(t)^2 + y'(t)^2 = 1, \quad \forall\ t \in [0, 2\pi].$

Let Ω denote the region bounded by Γ. To estimate the area of Ω, we use Green Thm in Calculus:

Green Thm :

$$\oint_{\Gamma} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

In particular, taking $Q(x,y) = x$ and $P(x,y) = 0$ gives

$$\oint_{\Gamma} x\,dy = \iint_{\Omega} 1\,dx\,dy = \text{Area}(\Omega) = A.$$

Notice that

$$\oint_{\Gamma} x\,dy = \int_0^{2\pi} x(t)\,y'(t)\,dt = A.$$

Let us expand $x(t)$, $y(t)$ into their Fourier series on $[0, 2\pi]$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \qquad y(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}.$$

$$x'(t) \sim \sum_{n=-\infty}^{\infty} i n a_n e^{int}, \quad y'(t) \sim \sum_{n=-\infty}^{\infty} i n b_n e^{int}.$$

By Parseval identity

$$1 = \frac{1}{2\pi}\int_0^{2\pi} x'(t)^2 + y'(t)^2\, dt = \sum_{n=-\infty}^{\infty}\left(|in a_n|^2 + |in b_n|^2\right)$$

$$= \sum_{n=-\infty}^{\infty} n^2\left(|a_n|^2 + |b_n|^2\right).$$

Moreover

$$\frac{1}{2\pi}\int_0^{2\pi} x(t)\, y'(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, \overline{y'(t)}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \hat{x}(n)\, \overline{\hat{y'}(n)}$$

$$= \sum_{n=-\infty}^{\infty} a_n \cdot \overline{in b_n}$$

Hence $\quad A = 2\pi \displaystyle\sum_{n=-\infty}^{\infty} a_n \cdot (-in)\, \overline{b_n}$

Hence

$$A = 2\pi \cdot \left| \sum_{n=-\infty}^{\infty} a_n (-in)\, \overline{b_n} \right|$$

$$\le 2\pi \sum_{n=-\infty}^{\infty} |n| |a_n| |b_n|$$

$$\le 2\pi \sum_{n=-\infty}^{\infty} |n| \frac{|a_n|^2 + |b_n|^2}{2}$$

$$\le 2\pi \cdot \sum_{n=-\infty}^{\infty} n^2 \frac{|a_n|^2 + |b_n|^2}{2} \le \frac{2\pi}{2} = \pi$$

This proves the isoperimetric inequality!

Suppose $A = \pi$. We must have

① $|a_n| = |b_n|$ if $n \neq 0$ $\left(\because |n|\,|a_n||b_n| = |n| \frac{|a_n|^2 + |b_n|^2}{2} \right)$

② $|a_n| = |b_n| = 0$ if $|n| > 1$. $\left(\because |n| \cdot \frac{|a_n|^2 + |b_n|^2}{2} = |n|^2 \cdot \frac{|a_n|^2 + |b_n|^2}{2} \right)$

Now

$$x(t) = a_{-1} e^{-it} + a_0 + a_1 e^{it}$$
$$y(t) = b_{-1} e^{-it} + b_0 + b_1 e^{it}$$

Since $x(t)$, $y(t)$ are real,

$$\overline{a_1} = a_{-1}, \qquad \overline{b_1} = b_{-1}$$

$$\overline{a_1} = \overline{\frac{1}{2\pi} \int_0^{2\pi} x(t) e^{-it} dt} = \frac{1}{2\pi} \int_0^{2\pi} \overline{x(t)}\, e^{it} dt$$
$$= \frac{1}{2\pi} \int_0^{2\pi} x(t) e^{it} dt$$
$$= a_{-1}$$

Hence $$|a_1| = |a_{-1}| = |b_1| = |b_{-1}|.$$

$$1 = \sum_{n=-\infty}^{\infty} |n|^2 (|a_n|^2 + |b_n|^2) = |a_1|^2 + |b_1|^2 + |a_{-1}|^2 + |b_{-1}|^2$$

$$\Rightarrow \quad |a_1| = |b_1| = |a_{-1}| = |b_{-1}| = \frac{1}{2}$$

So we can write

$$a_1 = \frac{1}{2} e^{i\alpha}$$

$$b_1 = \frac{1}{2} e^{i\beta}$$

for $\alpha, \beta \in [0, 2\pi]$.

Then $x(t) = a_{-1} e^{-it} + a_0 + a_1 e^{it}$

$$= \overline{a_1} e^{-it} + a_0 + a_1 e^{it}$$

$$= \frac{1}{2} e^{-i(\alpha+t)} + a_0 + \frac{1}{2} e^{i(\alpha+t)}$$

$$= a_0 + \cos(\alpha+t)$$

Similarly

$$y(t) = b_0 + \cos(\beta+t).$$

Recall

$$\pi = 2\pi \sum_{n\in\mathbb{Z}} a_n (-in) \overline{b_n}$$

$$= (-2\pi i) \cdot (a_1 \overline{b_1} - a_{-1} \overline{b_{-1}})$$

$$= (-2\pi i)\left(\frac{1}{4}e^{i(\alpha-\beta)} - \frac{1}{4}e^{i(\beta-\alpha)}\right)$$

$$= \pi \sin(\alpha-\beta)$$

Hence $\sin(\alpha-\beta) = 1$. So

$$\alpha - \beta = \pi/2 \quad \text{or} \quad -3\pi/2$$

So $$y(t) = b_0 + \cos(\beta+t)$$
$$= b_0 + \cos(\alpha+t-\pi/2)$$
$$= b_0 + \sin(\alpha+t).$$

i.e. $$\begin{cases} x(t) = a_0 + \cos(\alpha+t) \\ y(t) = b_0 + \sin(\alpha+t) \end{cases}$$

That means Γ is a circle.

$\blacksquare$.